Stock and Bond Valuation: Annuities and Perpetuities

Lecture Slides 3

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Important Shortcut Formulas

- Present value formula= main workshorse for valuing investments.
- Investment rarely have just 2 or 3 future payments:
  - Stocks may pay dividends forever.
  - Most commond mortgage bond: 30 years of monthly payments...
- NPV with 360 terms: not practical $\Rightarrow$ Use of shortcut formulas. When?

Perpetuities I

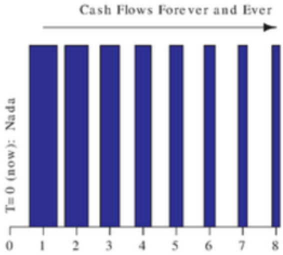
The Perpetuity Formula

- Definition: a project that has a stream of constant cash flows... repeated forever!
- Useful when:
  - Amount of money remains the same or grows at a constant rate.
  - The cost of capital/discount rate is constant.
- Useful for: coming up with quick rule-of-thumb estimates.

Perpetuities II

Interpretation

Time	Cash Flow	Discount Factor	Present Value	Cumul PV
0	Nothing! You have no cash flow here!			
1	\$2	$1/(1+10\%)^1 \approx 0.909$	\$1.82	\$1.82
2	\$2	$1/(1+10\%)^2 \approx 0.826$	\$1.65	\$3.47
3	\$2	$1/(1+10\%)^3 \approx 0.751$	\$1.50	\$4.97
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	\$2	$1/(1+10\%)^{50} \approx 0.0085$	\$0.02	\$19.83
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
			Net Present Value (Sum):	\$20.00



- Perpetuity $PV = \frac{\$2}{10\%} = \frac{\$2}{0.1} = \$20$; $PV = \frac{C_1}{r}$, a shortcut for:

$$PV = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \cdots + \frac{C_T}{(1+r)^T} + \cdots$$

Anecdote 3.1.a

The Oldest Institutions and Perpetuities

- Do projects last for ever?
    - Oldest Western Institution: Roman Catholic Church.
    - Oldest existing corporation in the US: 1628.

Questions 3.1.a

1. What is the PV of a perpetuity paying \$10 each month, beginning next month, if the monthly interest rate is constant 0.3%/month?
2. What is the PV of a perpetuity paying \$10 each month, beginning next month, if the **effective** interest rate is a constant 10% per year?
3. Under what interest rates would you prefer a perpetuity that pays \$1 million per year beginning next year to a one-time payment of \$20 million?

Perpetuities III

The Growing Perpetuity Formula

- What if the cash flows increase over time?
- As long as $g < r$:

$$PV\ of\ Growing\ Perpetuity = \frac{C_1}{r-g}$$

- Problems otherwise!

Why Growing Perpetuities?

TABLE 3.2 PERPETUITY STREAM WITH $C_1 = \$2$, GROWTH RATE $g = 5\%$, AND INTEREST RATE $r = 10\%$

Time	Cash Flow	Discount Rate	Discount Factor	Present Value	Cumulative
0	Nothing! You have no cash flow here!				
1	$(1+5\%)^0 \cdot \$2 = \$2.000$	$(1+10\%)^1$	0.909	\$1.818	\$1.82
2	$(1+5\%)^1 \cdot \$2 = \$2.100$	$(1+10\%)^2$	0.826	\$1.736	\$3.56
3	$(1+5\%)^2 \cdot \$2 = \$2.205$	$(1+10\%)^3$	0.751	\$1.657	\$5.22
$\vdots$					
30	$(1+5\%)^{29} \cdot \$2 \approx \$8.232$	$(1+10\%)^{30}$	0.057	\$0.472	\$30.09
$\vdots$					
			Net Present Value (Sum):		\$40.00

- Renting your appartment for ever (or 30 years): *inflation*!

Questions 3.1.b

1. What is the PV of a perpetuity paying \$10 each month, if the monthly interest rate is a constant 0.3%/month (3.6%/year) and the cash flows will grow at a rate of 0.1%/month (1.2%/year)?
2. An eternal patent contract states that the patentee will pay the patentor a fee of \$1 million next year. The contract terms state a fee growth with the inflation rate, which runs at 1.5% per annum. The appropriate cost of capital is 10%. What is the value of this patenting contract?
3. How would the patent contract value change if the first payment did not occur next year, but tonight?

Application 3.1.a

Stock Valuation with a Gordon Growth Model

- Applying it to the real world: approximation!!!

$$Business\ \ Value = \frac{\$1,000,000}{8\% - 2\%} \approx \$16,666,667$$

- Applying it to the stock market:

$$P\ \ Today = \frac{Dividends\ \ D\ \ Next\ \ Year}{r-g} \iff \frac{Dividends\ \ Next\ \ Year}{Stock\ \ Price\ \ Today} = r-g$$

Using the Gordon Model to value Stock

An example

- General Electric, 2003: dividend yield= 2.43% (Yahoo! Finances)

$$\frac{Dividends\ Next\ Year}{Stock\ Price\ Today} = r - g = 2.43\%$$

- GE paid \$7.643 billion in dividends in 2003 and \$6.358 billion in 2001 $\Rightarrow$

- Growth rate of dividends was about 9.6% per annum: $\$6.358 \cdot 1.096^2 \approx \$7.643 \Rightarrow$

- Assuming 9.6%/year is a fair representation of the eternal future growth rate of GE's dividends:

$$r = \frac{Dividends\ Next\ Year}{Stock\ Price\ Today} + g \approx 2.4\% + 9.6\% = 12\%$$

Thinking about the value of stocks

- As the value of the earnings stream the stocks will produce.
- Common to assume that stock market values are capitalized as if corporate earnings were eternal cash flows growing at a constant rate g applicable to earnings $\Rightarrow$
- Estimate the value of the firm as:

$$Stock\ Price\ P\ Today = \frac{Earnings\ E\ Next\ Year}{r - g}$$

- Trailing P/E and forward P/E:

$$r = \frac{Earnings\ Next\ Year}{Stock\ Price\ Today} + g = \frac{1}{P/E} + g \approx \frac{1}{18.5} + 6.3\% \approx 11.7\%$$

.

Questions 3.1.c

1. A stock is paying a quarterly dividend of \$10 in 1 month. The dividend is expected to increase every quarter by the inflation rate of 0.3% per quarter - so it will be %10.03 next quarter-. The prevailing cost of capital for this kind of stock is 10% per annum. What should this stock be worth?
2. If a \$50 stock has earnings of \$3 per year, and the appropriate cost of capital for this stock is 12% per year, what does the market expect the firm's "as-if-eternal dividends" to grow at?

Annuities I

Definition and formula

- Definition: A stream of equal cash flows for a given number of periods T, discounted at a constant interest rate r.

$$PV = \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_2)} + \frac{C_3}{(1+r_3)}$$

- The annuity formula:

$$PV = \frac{C_1}{r} \cdot \left\{ 1 - \frac{1}{(1+r)^T} \right\}$$

Questions 3.1.d

1. How many years does it take for an annuity to reach half of the value of a perpetuity if the interest rate is 3%? If the interest rate is r?
2. What is the PV of a 120-month annuity paying \$10 per month, beginning at \$10 next month (time 1), if the monthly interest rate is a constant 0.3% /month (3.6%/year)?

Anecdote 3.1.b

Fibonacci and the Invention of Net Present Value

- Fibonacci (Leonardo of Pisa), might have invented the concept of NPV.
- Family: merchants in the Mediterranean in the 13th century.
- He wrote about mathematics mainly as a tool to solve merchant's economic problems.

Application 3.1.b

Fixed-Rate Mortgage Payments

- Fixed-rate mortgage loans are annuities: promise equal cash payments each month to a lender.
- 30-year mortgage with monthly payments = 360-payment annuity.
- Mortgage providers: quote interest by just dividing the mortgage quota by 12:
  - $r = 7.5\% \Rightarrow \textit{monthly } r = 7.5\%/12 = 0.625\%.$

$$PV = \frac{C_1}{r} \cdot \left[1 - \frac{1}{(1+r)^T} \right]$$

- Where C_1 is the monthly payment.

Principal and Interest Components

- Reasons to distinguish these two elements:
  - You need to know how much principal you owe if you want to repay the loan early.
  - The government (US) allows mortgage borrowers to deduct the interest, but not the principal, from their tax bills.

> **SIDE NOTE:** Uncle Sam allows mortgage borrowers to deduct the interest, but not the principal, from their tax bills. The IRS imputes interest on the above mortgage as follows: In the first month, Uncle Sam proclaims $0.625\% \cdot \$500{,}000 = \$3{,}125$ to be the tax-deductible mortgage interest payment. Therefore, the principal repayment is $\$3{,}496.07 - \$3{,}125 = \$371.07$ and the remaining principal is \$499,628.93. The following month, Uncle Sam proclaims $0.625\% \cdot \$499{,}628.93 \approx \$3{,}122.68$ to be the tax-deductible interest payment, $\$3{,}496.07 - \$3{,}122.68 = \$373.39$ to be the principal repayment, and \$499,255.54 as the remaining principal. And so on.

Questions 3.1.e

Annuities

1. What would your rate of return be if you rented your \$250,000 warehouse for 10 years at at monthly lease payment of \$2,500? If you can earn 5% per annum elsewhere, would you rent your warehouse? (Rental agreements $\Longleftrightarrow$ mortgages).
2. What is the monthly payment on a 10-year mortgage for every \$1,000 of mortgage at an effective interest rate of 6% per year?

Application 3.1.c

A Level-Coupon Bond

- Zero -coupon- bonds: one payment at maturity (principal + interest).
- Coupon bonds: Pays cash at many different points in time.
- Level-coupon bonds: coupon payments that remain the same for the life of the bond.

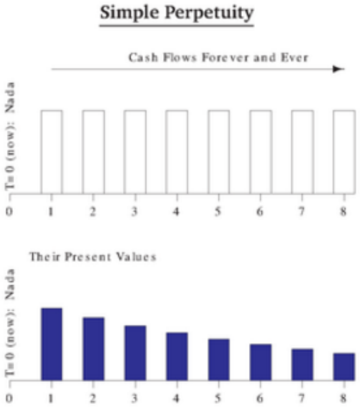
Example of a level-coupon bond

- A bond that pays \$1,500 twice a year for 5 years + an additional \$100,000 in 5 years.
- Called a "3% semiannual coupon bond": $1500 * 2 = 3000,\ 3000/100000 = 0.03 = 3\%$.
- This 3% is different from the interest rate! Pricing this bond:

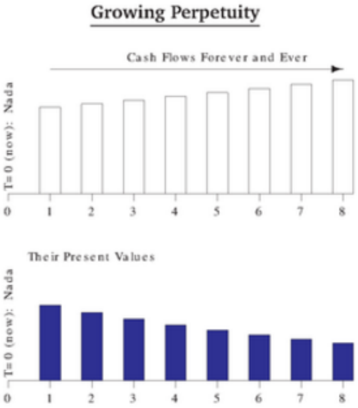
Year	Due Date	Bond Payment	Rate of Return	Discount Factor	Present Value
0.5	Nov 2002	\$1,500	2.47%	0.9759	\$1,463.85
1.0	May 2003	\$1,500	5.00%	0.9524	\$1,428.57
1.5	Nov 2003	\$1,500	7.59%	0.9294	\$1,394.14
2.0	May 2004	\$1,500	10.25%	0.9070	\$1,360.54
2.5	Nov 2004	\$1,500	12.97%	0.8852	\$1,327.76
3.0	May 2005	\$1,500	15.76%	0.8638	\$1,295.76
3.5	Nov 2005	\$1,500	18.62%	0.8430	\$1,264.53
4.0	May 2006	\$1,500	21.55%	0.8277	\$1,234.05
4.5	Nov 2006	\$1,500	24.55%	0.8029	\$1,204.31
5.0	May 2007	\$101,500	27.63%	0.7835	\$79,527.91
				Sum:	\$91,501.42

The Formulas Summarized I

Perpetuities



Formula: $PV = \frac{CF}{r}$

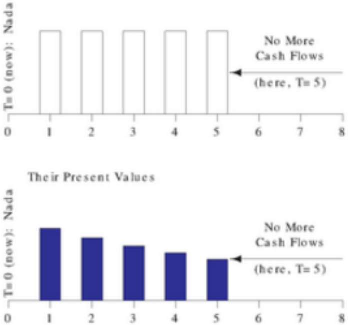


Formula: $PV = \frac{CF_1}{r-g}$

The Formulas Summarized II

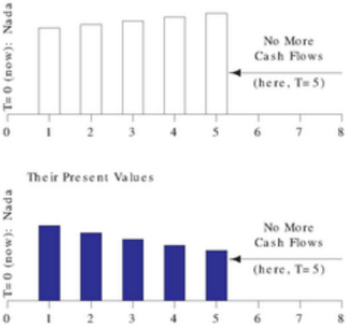
Annuities

Simple Annuity



Formula: $PV = \frac{CF}{r} \cdot \left[1 - \left(\frac{1}{1+r}\right)^T\right]$

Growing Annuity



Formula: $PV = \frac{CF_1}{r-g} \cdot \left[1 - \left(\frac{1+g}{1+r}\right)^T\right]$

Summary

- PV of a simple perpetuity.
- PV of a growing perpetuity.
- The Gordon dividend growth model: valuing stocks through the growing perpetuity formula.
- PV of an annuity.
- Fixed-rate mortgages are annuities.

Keywords

Annuity, coupon bond, discount, dividend yield, fixed-rate mortgage, Gordon growth model, growing annuity, growing perpetuity, level-coupon bond, perpetuity, premium, principal, zero-bond.

Exercizes for next lecture

- Study for midterm exam.
- Quiz 5.